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MATHEMATICS
9740/02
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1 (i)

$x^2 - 6x + 34 = 0$

$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(34)}}{2}$

$= \dfrac{6 \pm \sqrt{-100}}{2}$

$= \dfrac{6 \pm 10\sqrt{-1}}{2}$

$= \dfrac{6 \pm 10\mathrm{i}}{2}$

$= 3 \pm 5\mathrm{i}$

(ii) Let $f(x) = x^4 + 4x^3 + x^2 + ax + b$

Since $-2+\mathrm{i}$ is a root of the equation, using the factor theorem,

$f(-2+\mathrm{i})$

$= (-2+\mathrm{i})^4 + 4(-2+\mathrm{i})^3 + (-2+\mathrm{i})^2 + a(-2+\mathrm{i}) + b$

$= -12 + 16\mathrm{i} - 2a + a\mathrm{i} + b$

$= -12 - 2a + b + 16\mathrm{i} + a\mathrm{i}$

$= 0$

$16 + a = 0$ $\qquad$ $-12 - 2a + b = 0$

$a = -16$ $\qquad$ $b = 12 + 2(-16)$

$= 12 - 32$

$= -20$

Since $-2+\mathrm{i}$ is a root of the equation, $-2-\mathrm{i}$ is also a root of the equation.

Given $a = -2$ and $b = 1$,

$x^2 - 2ax + a^2 + b^2$

$= x^2 - 2(-2)x + \left[(-2)^2 + (1)^2\right]$

$= x^2 + 4x + 4 + 1$

$= x^2 + 4x + 5$

$x^4 + 4x^3 + x^2 - 16x - 20$

$= (x^2 + 4x + 5)(x^2 - 4)$

$= (x^2 + 4x + 5)(x+2)(x-2)$

$= 0$

The other roots are $2, -2$ and $-2-\mathrm{i}$.

2 (i) Let P_n be the statement

$$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$$

When $n = 1$,

LHS $= 1(1+2)$ $\qquad$ RHS $= \frac{1}{6}(1)(1+1)(2(1)+7)$

$= 3$ $\qquad$ $= \frac{1}{6}(1)(2)(9)$

$= 3$

$\therefore$ Since LHS = RHS, P_1 is true.

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$$

To show that P_{k+1} is also true, i.e.

$$\sum_{r=1}^{k+1} r(r+2) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+7)$$

$$= \frac{1}{6}(k+1)(k+3)(2k+9)$$

$$\sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^{k+1} r(r+2) + (k+1)((k+1)+2)$$

$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

$$= \frac{1}{6}(k+1)\left[k(2k+7) + 6(k+3)\right]$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18)$$

$$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$$

$$= \frac{1}{6}(k+1)(k+3)(2k+9)$$

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for all $n \in \mathbb{Z}^+$ by mathematical induction.

(ii) (a) $\dfrac{1}{r(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+2} \Rightarrow 1 = A(r+2) + B(r)$

When $r = 0$,

$2A = 1 \Rightarrow A = \frac{1}{2}$

When $r = -2$,

$-2B = 1 \Rightarrow B = -\frac{1}{2}$

$\therefore \dfrac{1}{r(r+2)} = \dfrac{1}{2r} - \dfrac{1}{2(r+2)}$

$$\sum_{r=1}^{n}\frac{1}{r(r+2)}=\sum_{r=1}^{n}\frac{1}{2r}-\frac{1}{2(r+2)}$$

$$\begin{aligned}
&=\tfrac{1}{2}-\cancel{\tfrac{1}{6}}\\
&+\tfrac{1}{4}-\cancel{\tfrac{1}{8}}\\
&+\cancel{\tfrac{1}{6}}-\cancel{\tfrac{1}{10}}\\
&\vdots\\
&+\cancel{\tfrac{1}{2(n-2)}}-\cancel{\tfrac{1}{2n}}\\
&+\cancel{\tfrac{1}{2(n-1)}}-\tfrac{1}{2(n+1)}\\
&+\cancel{\tfrac{1}{2n}}-\tfrac{1}{2(n+2)}\\
&=\frac{1}{2}+\frac{1}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}\\
&=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}
\end{aligned}$$

(b) Since $\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$,

as $n\to\infty$, $\frac{1}{2(n+1)}\to 0$ and $\frac{1}{2(n+1)}\to 0$

$\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{3}{4}-0-0=\frac{3}{4}$

Since it converges to a constant value, this is a convergent series.

The sum to infinity is $\frac{3}{4}$.

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3 **(i)** $y=x\sqrt{x+2}$

$$\frac{dy}{dx}=\frac{x}{2\sqrt{x+2}}+\sqrt{x+2}=\frac{x+2(x+2)}{2\sqrt{x+2}}=\frac{3x+4}{2\sqrt{x+2}}$$

If only one value of x satisfies the condition,

$$\frac{dy}{dx}=\frac{3x+4}{2\sqrt{x+2}}=0$$

$\Rightarrow 3x+4=0$

$\therefore$ There is only one value of x, i.e. $x=-\frac{4}{3}$.

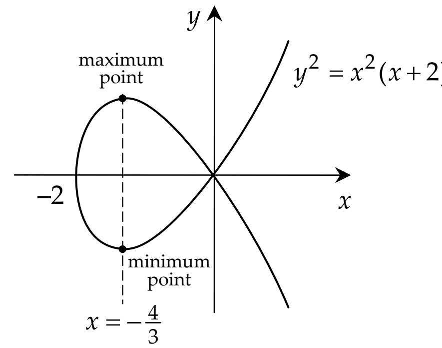
(ii) **(a)** $y^2=x^2(x+2)$

$$y=\pm\sqrt{x^2(x+2)}=\pm x\sqrt{x+2}$$

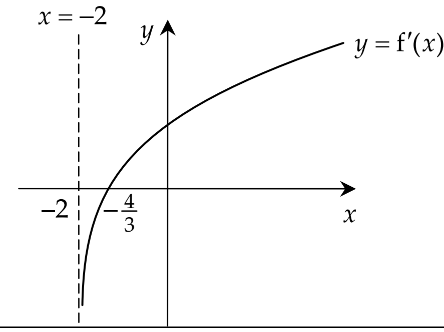
$$\frac{dy}{dx}=\pm\frac{3x+4}{2\sqrt{x+2}}=\pm\frac{4}{2\sqrt{2}}=\pm\frac{2}{\sqrt{2}}=\pm\sqrt{2}$$

The possible values of the gradient is $\sqrt{2}$ and $-\sqrt{2}$.

(b)



(iii)



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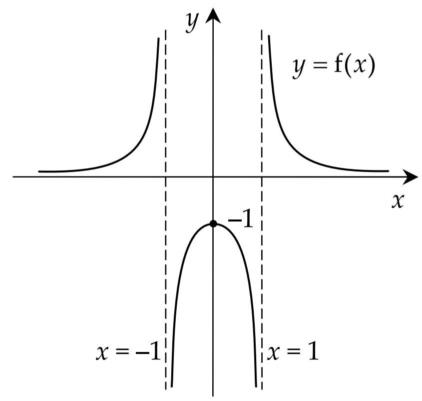
4 **(i)**



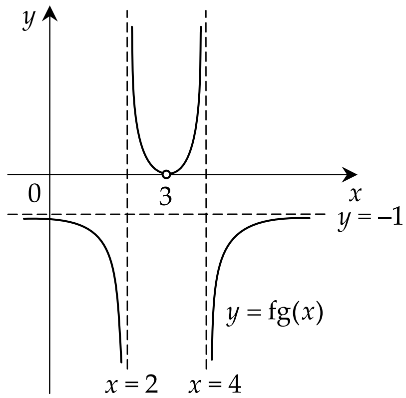
(ii) The least value of k is 0, such that the function is one-one, where each element has a unique image. When the function is one-one, its inverse exists.

(iii)

$$\frac{1}{\left(\frac{1}{x-3}\right)^2-1}=\frac{1}{\frac{1-(x-3)^2}{(x-3)^2}}=\frac{(x-3)^2}{1-(x-3)^2}=\frac{(x-3)^2}{1-x^2+6x-9}=\frac{(x-3)^2}{-x^2+6x-8}=\frac{(x-3)^2}{(4-x)(x-2)}$$

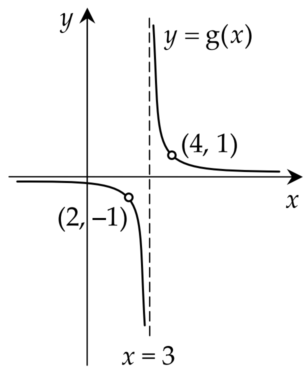
(iv) To solve $\frac{(x-3)^2}{(4-x)(x-2)}>0$, sketch the graph of

$$y=\frac{(x-3)^2}{(4-x)(x-2)}.$$



$\therefore 2 < x < 3$ or $3 < x < 4$

(v)



Since $x \neq 2,\ x \neq 3$ and $x \neq 4$,

$R_g = (-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty)$.

We obtain R_{fg} based on R_g being put on the x-axis on the graph in **(i)**.

$\therefore R_{fg} = (-\infty,-1)\cup(0,\infty)$

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5 **(i)** It is difficult to divide the spectators into appropriate strata and to identify and classify every member, given the large number of spectators.

(ii) Number the spectators from 1 to N based on their ticket serial numbers. To select a sample size of $0.01N$, we choose a random spectator from the first $\frac{N}{0.01N} = 100$ spectator, then every 100th spectator.

6

$$\bar{t} = \frac{\Sigma t}{n} = \frac{454.3}{11} = 41.3$$

$$s^2 = \frac{1}{n-1}\left(\Sigma t^2 - \frac{(\Sigma t)^2}{n}\right)$$

$$= \frac{1}{11-1}\left(18778.43 - \frac{454.3^2}{11}\right)$$

$$= 1.584$$

The unbiased estimates of the population mean and variance are 41.3 and 1.584 respectively.

Let X be the mean time in minutes required by an employee to complete a task.

$H_0 : \mu = 42.0$ vs $H_1 : \mu \neq 42.0$

Perform a 2-tail test at the 10% significance level.

Under H_0, $T = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$, where

$\mu_0 = 42.0,\ \bar{t} = 41.3,\ s^2 = 1.584$ and $n = 11$.

Using a t-test, p-value = 0.0949.

Since the p-value = 0.0949 < 0.1, we reject H_0 and conclude that there is sufficient evidence at the 10% significance level that the mean time required by an employee to complete a task has changed.

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7 **(i)**

$$P(A \cap B') = P(A \mid B') \times P(B')$$
$$= P(A \mid B') \times (1 - P(B))$$
$$= 0.8 \times (1 - 0.6)$$
$$= 0.32$$

(ii)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - (P(A) - P(A \cap B'))$$
$$= P(A) + P(B) - P(A) + P(A \cap B')$$
$$= P(B) + P(A \cap B')$$
$$= 0.6 + 0.32$$
$$= 0.92$$

(iii)

$$P(B' \mid A) = \frac{P(B' \cap A)}{P(A)}$$
$$= \frac{0.32}{0.7}$$
$$= \frac{16}{35}$$

(iv) Since A and C are independent,

$$P(A \cap C) = P(A) \times P(C)$$
$$= 0.7 \times 0.5$$
$$= 0.35$$
$$P(A' \cap C) = P(C) - P(A \cap C)$$
$$= 0.5 - 0.35$$
$$= 0.15$$

(v) Given $P(A \cup B) = 0.92$,

$$P((A \cup B)') = 1 - P(A \cup B)$$
$$= 1 - 0.92$$
$$= 0.08$$
$$P((A \cup B)' \cap C) \leq 0.08$$
$$P(A' \cap B' \cap C) \leq 0.08$$
$$P(A' \cap B \cap C) = P(A' \cap C) - P(A' \cap B' \cap C)$$
$$\geq 0.15 - 0.08$$
$$\geq 0.07$$

$$\therefore 0.07 \leq P(A' \cap B \cap C) \leq 0.15$$

8 (i) P(number is greater than 30 000)

$= \dfrac{\text{n(number is greater than 30 000)}}{\text{n(total number of arrangements)}}$

$= \dfrac{3\times 4!}{5!}$

$= \dfrac{3}{5}$

(ii) P(last two digits are both even)

$= \dfrac{\text{n(last two digits are both even)}}{\text{n(total number of arrangements)}}$

$= \dfrac{2!3!}{5!}$

$= \dfrac{1}{10}$

(iii) Case 1 : The last digit is 1

Number of arrangements

$=1\times 3\times 3!$

$= 18$

Case 2 : The last digit is 3 or 5

Number of arrangements

$=2\times 2\times 3!$

$= 24$

P(number is greater than 30 000 and odd)

$= \dfrac{\text{n(last two digits are both even)}}{\text{n(total number of arrangements)}}$

$= \dfrac{18+24}{5!}$

$= \dfrac{7}{20}$

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9 (i) Given $X \sim \text{N}(180, 30^2)$ and $Y \sim \text{N}(400, 60^2)$,

$Y - 2X \sim \text{N}(400 - 2(180), 60^2 + 2^2\cdot 30^2)$

$Y - 2X \sim \text{N}(40, 7200)$

$\text{P}(Y > 2X) = \text{P}(Y - 2X > 0)$

$= 0.68132$

≈ 0.681

(ii)

$0.12X + 0.05Y \sim \text{N}\begin{pmatrix} 0.12(180)+0.05(400), \\ 0.12^2\cdot 30^2 + 0.05^2\cdot 60^2 \end{pmatrix}$

$0.12X + 0.05Y \sim \text{N}(41.6, 21.96)$

$\text{P}(0.12X + 0.05Y > 45)$

$= 0.23406$

≈ 0.234

(iii)

$0.12(X_1 + X_2) \sim \text{N}\begin{pmatrix} 0.12(180+180), \\ 0.12^2(30^2+30^2) \end{pmatrix}$

$0.12(X_1 + X_2) \sim \text{N}(43.2, 25.92)$

$\text{P}(0.12(X_1 + X_2) > 45)$

$= 0.36184$

≈ 0.362

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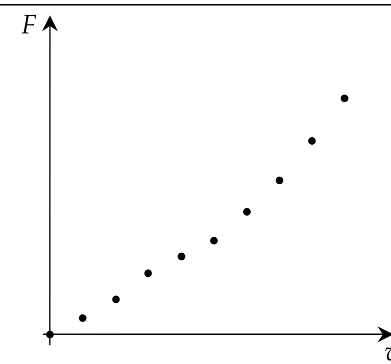
10 (i)



(ii) (a) Product moment correlation coefficient between v and $F = 0.9860$

(b) Product moment correlation coefficient between v^2 and $F = 0.9907$

(iii) $F = c + dv^2$ gives the better model since the product moment correlation coefficient is closer to 1.

(iv) Using GC, the equation of a suitable regression line is $F = 3.20 + 0.0242v^2$.

When $F = 26$,

$26 = 3.20 + 0.0242v^2$

$\Rightarrow v = \pm 30.7$

Since $v \geq 0$, $v = 30.7$

Since the wind speed v is the independent variable and the drag force F is the dependent variable, we should find the required estimate only when the horizontal axis is expressed in terms of v.

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11 (i) Let X be the number of telephone calls received by a call centre in one minute, i.e. $X \sim \text{Po}(3)$.

Using the additive property, $4X \sim \text{Po}(12)$.

$\text{P}(4X = 8) = 0.065523$

≈ 0.0655

(ii) Let n denote the number of seconds for which no calls are received.

$\text{Mean} = \dfrac{3n}{60} = 0.05n$

Using the additive property, $nX \sim \text{Po}(0.05n)$

$\text{P}(nX = 0) = 0.2$

$e^{-0.05n}\dfrac{(0.05n)^0}{0!} = 0.2$

$e^{-0.05n} = 0.2$

$-0.05n = \ln 0.2$

$n = \dfrac{\ln 0.2}{-0.05}$

$= 32.189$

≈ 32

(iii) Number of minutes $= 12 \times 60 = 720$

Using the additive property, $720X \sim \text{Po}(2160)$

Since $\lambda = 2160$ is large, we can use a normal distribution to approximate the Poisson distribution, where $720X \sim \text{N}(2160, 2160)$ approximately.

$\text{P}(720X > 2200)$

$= \text{P}(720X > 2200.5)$ (by continuity correction)

$= 0.19176$

≈ 0.192

(iv) Let Y be the number of busy working days out of 6 working days.

$Y \sim \text{B}(6, 0.192)$

$\text{P}(Y = 2) = 0.23538$

≈ 0.235

(v) Let W be the number of busy working days out of 30 working days.

$np = 30 \times 0.19176 \approx 5.75 > 5$

$n(1-p) = 30 \times (1 - 0.19176) \approx 24.2 > 5$

Since $np > 5$ and $n(1-p) > 5,$ we use a normal distribution to approximate the binomial distribution.

$np(1-p) = 30 \times 0.19176 \times (1 - 0.19176) = 4.65$

$W \sim \text{N}(5.75, 4.65)$

$\text{P}(W < 10)$

$= \text{P}(W < 9.5)$ (by continuity correction)

$= 0.95888$

≈ 0.959