

1 (i)  $x^2 - 6x + 34 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(34)}}{2}$$

$$= \frac{6 \pm \sqrt{-100}}{2}$$

$$= \frac{6 \pm 10\sqrt{-1}}{2}$$

$$= \frac{6 \pm 10i}{2}$$

$$= 3 \pm 5i$$

(ii) Let  $f(x) = x^4 + 4x^3 + x^2 + ax + b$

Since  $-2 + i$  is a root of the equation, using the factor theorem,

$$f(-2 + i)$$

$$= (-2 + i)^4 + 4(-2 + i)^3 + (-2 + i)^2 + a(-2 + i) + b$$

$$= -12 + 16i - 2a + ai + b$$

$$= -12 - 2a + b + 16i + ai$$

$$= 0$$

$$16 + a = 0 \quad -12 - 2a + b = 0$$

$$a = -16 \quad b = 12 + 2(-16)$$

$$= 12 - 32$$

$$= -20$$

Since  $-2 + i$  is a root of the equation,  $-2 - i$  is also a root of the equation.

Given  $a = -2$  and  $b = 1$ ,

$$x^2 - 2ax + a^2 + b^2$$

$$= x^2 - 2(-2)x + [(-2)^2 + (1)^2]$$

$$= x^2 + 4x + 4 + 1$$

$$= x^2 + 4x + 5$$

$$x^4 + 4x^3 + x^2 - 16x - 20$$

$$= (x^2 + 4x + 5)(x^2 - 4)$$

$$= (x^2 + 4x + 5)(x + 2)(x - 2)$$

$$= 0$$

The other roots are 2,  $-2$  and  $-2 - i$ .

2 (i) Let  $P_n$  be the statement

$$\sum_{r=1}^n r(r+2) = \frac{1}{6}n(n+1)(2n+7)$$

When  $n = 1$ ,

$$\text{LHS} = 1(1+2) = 3$$

$$\text{RHS} = \frac{1}{6}(1)(1+1)(2(1)+7)$$

$$= \frac{1}{6}(1)(2)(9)$$

$$= 3$$

$\therefore$  Since LHS = RHS,  $P_1$  is true.

Assume that  $P_k$  is true for some  $k \in \mathbb{Z}^+$ , i.e.

$$\sum_{r=1}^k r(r+2) = \frac{1}{6}k(k+1)(2k+7)$$

To show that  $P_{k+1}$  is also true, i.e.

$$\sum_{r=1}^{k+1} r(r+2) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+7)$$

$$= \frac{1}{6}(k+1)(k+3)(2k+9)$$

$$\sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^k r(r+2) + (k+1)((k+1)+2)$$

$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

$$= \frac{1}{6}(k+1)[k(2k+7) + 6(k+3)]$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18)$$

$$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$$

$$= \frac{1}{6}(k+1)(k+3)(2k+9)$$

Since  $P_1$  is true and  $P_{k+1}$  is true whenever  $P_k$  is true,  $P_n$  is true for all  $n \in \mathbb{Z}^+$  by mathematical induction.

(ii) (a)  $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2} \Rightarrow 1 = A(r+2) + B(r)$

When  $r = 0$ ,

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

When  $r = -2$ ,

$$-2B = 1 \Rightarrow B = -\frac{1}{2}$$

$$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

$$\begin{aligned} \sum_{r=1}^n \frac{1}{r(r+2)} &= \sum_{r=1}^n \frac{1}{2r} - \frac{1}{2(r+2)} \\ &= \frac{1}{2} - \frac{1}{6} \\ &\quad + \frac{1}{4} - \frac{1}{8} \\ &\quad + \frac{1}{6} - \frac{1}{10} \\ &\quad \vdots \\ &\quad + \frac{1}{2(n-2)} - \frac{1}{2n} \\ &\quad + \frac{1}{2(n-1)} - \frac{1}{2(n+1)} \\ &\quad + \frac{1}{2n} - \frac{1}{2(n+2)} \\ &= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\ &= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \end{aligned}$$

(b) Since  $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$ ,

as  $n \rightarrow \infty$ ,  $\frac{1}{2(n+1)} \rightarrow 0$  and  $\frac{1}{2(n+2)} \rightarrow 0$

$$\therefore \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3}{4} - 0 - 0 = \frac{3}{4}$$

Since it converges to a constant value, this is a convergent series.

The sum to infinity is  $\frac{3}{4}$ .

3 (i)  $y = x\sqrt{x+2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{2\sqrt{x+2}} + \sqrt{x+2} \\ &= \frac{x+2(x+2)}{2\sqrt{x+2}} \\ &= \frac{3x+4}{2\sqrt{x+2}} \end{aligned}$$

If only one value of  $x$  satisfies the condition,

$$\begin{aligned} \frac{dy}{dx} = \frac{3x+4}{2\sqrt{x+2}} &= 0 \\ \Rightarrow 3x+4 &= 0 \end{aligned}$$

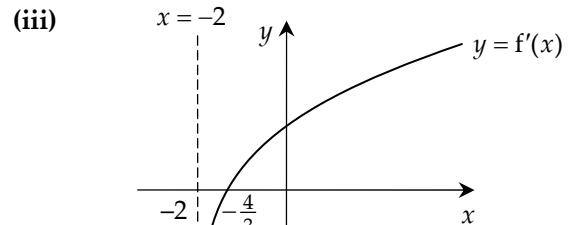
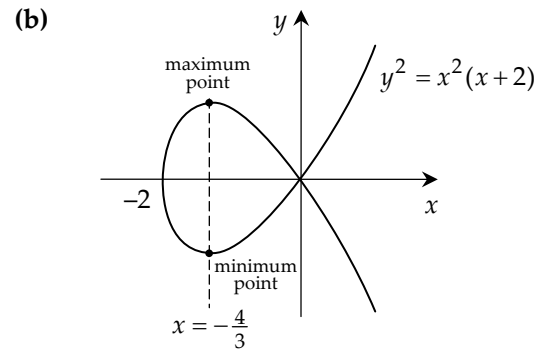
$\therefore$  There is only one value of  $x$ , i.e.  $x = -\frac{4}{3}$ .

(ii) (a)  $y^2 = x^2(x+2)$

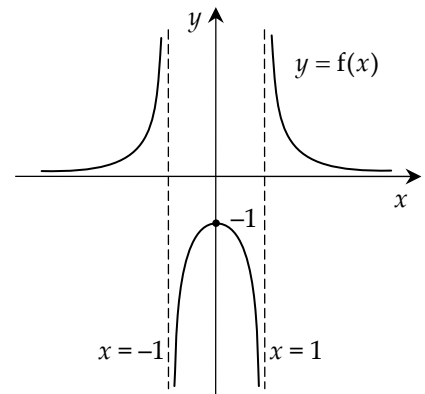
$$\begin{aligned} y &= \pm\sqrt{x^2(x+2)} \\ &= \pm x\sqrt{x+2} \end{aligned}$$

$$\frac{dy}{dx} = \pm \frac{3x+4}{2\sqrt{x+2}} = \pm \frac{4}{2\sqrt{2}} = \pm \frac{2}{\sqrt{2}} = \pm\sqrt{2}$$

The possible values of the gradient is  $\sqrt{2}$  and  $-\sqrt{2}$ .



4 (i)



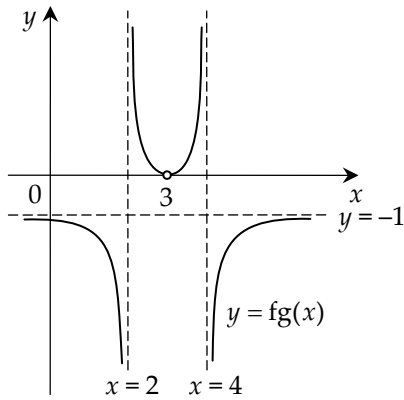
(ii) The least value of  $k$  is 0, such that the function is one-one, where each element has a unique image. When the function is one-one, its inverse exists.

(iii)

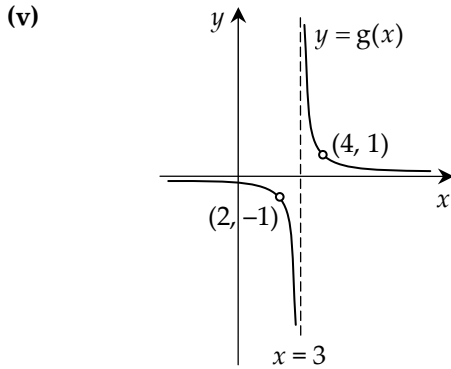
$$\begin{aligned} \frac{1}{\left(\frac{1}{x-3}\right)^2 - 1} &= \frac{1}{\frac{1-(x-3)^2}{(x-3)^2}} \\ &= \frac{(x-3)^2}{1-(x-3)^2} \\ &= \frac{(x-3)^2}{1-x^2+6x-9} \\ &= \frac{(x-3)^2}{-x^2+6x-8} \\ &= \frac{(x-3)^2}{(4-x)(x-2)} \end{aligned}$$

(iv) To solve  $\frac{(x-3)^2}{(4-x)(x-2)} > 0$ , sketch the graph of

$$y = \frac{(x-3)^2}{(4-x)(x-2)}$$



$\therefore 2 < x < 3$  or  $3 < x < 4$



Since  $x \neq 2$ ,  $x \neq 3$  and  $x \neq 4$ ,  
 $R_g = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$ .

We obtain  $R_{fg}$  based on  $R_g$  being put on the  $x$ -axis on the graph in (i).

$\therefore R_{fg} = (-\infty, -1) \cup (0, \infty)$

- 5 (i) It is difficult to divide the spectators into appropriate strata and to identify and classify every member, given the large number of spectators.
- (ii) Number the spectators from 1 to  $N$  based on their ticket serial numbers. To select a sample size of  $0.01N$ , we choose a random spectator from the first  $\frac{N}{0.01N} = 100$  spectator, then every 100th spectator.

6

$$\bar{t} = \frac{\sum t}{n} = \frac{454.3}{11} = 41.3$$

$$s^2 = \frac{1}{n-1} \left( \sum t^2 - \frac{(\sum t)^2}{n} \right)$$

$$= \frac{1}{11-1} \left( 18778.43 - \frac{454.3^2}{11} \right)$$

$$= 1.584$$

The unbiased estimates of the population mean and variance are 41.3 and 1.584 respectively.

Let  $X$  be the mean time in minutes required by an employee to complete a task.

$$H_0 : \mu = 42.0 \quad \text{vs} \quad H_1 : \mu \neq 42.0$$

Perform a 2-tail test at the 10% significance level.

Under  $H_0$ ,  $T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t(n-1)$ , where

$$\mu_0 = 42.0, \quad \bar{t} = 41.3, \quad s^2 = 1.584 \quad \text{and} \quad n = 11.$$

Using a  $t$ -test,  $p$ -value = 0.0949.

Since the  $p$ -value = 0.0949 < 0.1, we reject  $H_0$  and conclude that there is sufficient evidence at the 10% significance level that the mean time required by an employee to complete a task has changed.

- 7 (i)  $P(A \cap B') = P(A | B') \times P(B')$   
 $= P(A | B') \times (1 - P(B))$   
 $= 0.8 \times (1 - 0.6)$   
 $= 0.32$
- (ii)  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$   
 $= P(A) + P(B) - (P(A) - P(A \cap B'))$   
 $= P(A) + P(B) - P(A) + P(A \cap B')$   
 $= P(B) + P(A \cap B')$   
 $= 0.6 + 0.32$   
 $= 0.92$
- (iii)  $P(B' | A) = \frac{P(B' \cap A)}{P(A)}$   
 $= \frac{0.32}{0.7}$   
 $= \frac{16}{35}$
- (iv) Since  $A$  and  $C$  are independent,  
 $P(A \cap C) = P(A) \times P(C)$   
 $= 0.7 \times 0.5$   
 $= 0.35$   
 $P(A' \cap C) = P(C) - P(A \cap C)$   
 $= 0.5 - 0.35$   
 $= 0.15$
- (v) Given  $P(A \cup B) = 0.92$ ,  
 $P((A \cup B)') = 1 - P(A \cup B)$   
 $= 1 - 0.92$   
 $= 0.08$   
 $P((A \cup B)' \cap C) \leq 0.08$   
 $P(A' \cap B' \cap C) \leq 0.08$   
 $P(A' \cap B \cap C) = P(A' \cap C) - P(A' \cap B' \cap C)$   
 $\geq 0.15 - 0.08$   
 $\geq 0.07$   
 $\therefore 0.07 \leq P(A' \cap B \cap C) \leq 0.15$

8 (i)  $P(\text{number is greater than } 30\,000)$   
 $= \frac{n(\text{number is greater than } 30\,000)}{n(\text{total number of arrangements})}$   
 $= \frac{3 \times 4!}{5!}$   
 $= \frac{3}{5}$

(ii)  $P(\text{last two digits are both even})$   
 $= \frac{n(\text{last two digits are both even})}{n(\text{total number of arrangements})}$   
 $= \frac{2!3!}{5!}$   
 $= \frac{1}{10}$

(iii) Case 1: The last digit is 1  
 Number of arrangements  
 $= 1 \times 3 \times 3!$   
 $= 18$

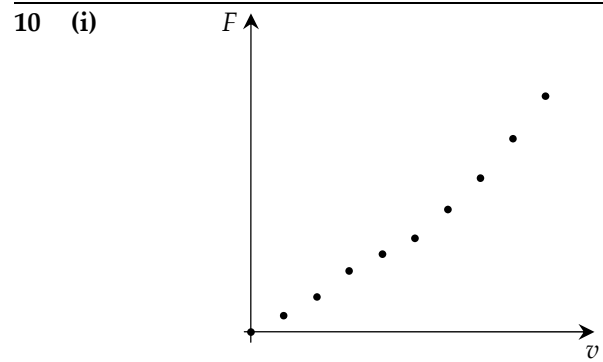
Case 2: The last digit is 3 or 5  
 Number of arrangements  
 $= 2 \times 2 \times 3!$   
 $= 24$

$P(\text{number is greater than } 30\,000 \text{ and odd})$   
 $= \frac{n(\text{last two digits are both even})}{n(\text{total number of arrangements})}$   
 $= \frac{18 + 24}{5!}$   
 $= \frac{7}{20}$

9 (i) Given  $X \sim N(180, 30^2)$  and  $Y \sim N(400, 60^2)$ ,  
 $Y - 2X \sim N(400 - 2(180), 60^2 + 2^2 \cdot 30^2)$   
 $Y - 2X \sim N(40, 7200)$   
 $P(Y > 2X) = P(Y - 2X > 0)$   
 $= 0.68132$   
 $\approx 0.681$

(ii)  $0.12X + 0.05Y \sim N\left(\frac{0.12(180) + 0.05(400)}{0.12^2 \cdot 30^2 + 0.05^2 \cdot 60^2}\right)$   
 $0.12X + 0.05Y \sim N(41.6, 21.96)$   
 $P(0.12X + 0.05Y > 45)$   
 $= 0.23406$   
 $\approx 0.234$

(iii)  $0.12(X_1 + X_2) \sim N\left(\frac{0.12(180 + 180)}{0.12^2(30^2 + 30^2)}\right)$   
 $0.12(X_1 + X_2) \sim N(43.2, 25.92)$   
 $P(0.12(X_1 + X_2) > 45)$   
 $= 0.36184$   
 $\approx 0.362$



- (ii) (a) Product moment correlation coefficient between  $v$  and  $F = 0.9860$   
 (b) Product moment correlation coefficient between  $v^2$  and  $F = 0.9907$
- (iii)  $F = c + dv^2$  gives the better model since the product moment correlation coefficient is closer to 1.
- (iv) Using GC, the equation of a suitable regression line is  $F = 3.20 + 0.0242v^2$ .  
 When  $F = 26$ ,  
 $26 = 3.20 + 0.0242v^2$   
 $\Rightarrow v = \pm 30.7$   
 Since  $v \geq 0$ ,  $v = 30.7$

Since the wind speed  $v$  is the independent variable and the drag force  $F$  is the dependent variable, we should find the required estimate only when the horizontal axis is expressed in terms of  $v$ .

- 11 (i) Let  $X$  be the number of telephone calls received by a call centre in one minute, i.e.  $X \sim \text{Po}(3)$ .  
 Using the additive property,  $4X \sim \text{Po}(12)$ .  
 $P(4X = 8) = 0.065523$   
 $\approx 0.0655$
- (ii) Let  $n$  denote the number of seconds for which no calls are received.  
 Mean  $= \frac{3n}{60} = 0.05n$   
 Using the additive property,  $nX \sim \text{Po}(0.05n)$   
 $P(nX = 0) = 0.2$   
 $e^{-0.05n} \frac{(0.05n)^0}{0!} = 0.2$   
 $e^{-0.05n} = 0.2$   
 $-0.05n = \ln 0.2$   
 $n = \frac{\ln 0.2}{-0.05}$   
 $= 32.189$   
 $\approx 32$

(iii) Number of minutes =  $12 \times 60 = 720$

Using the additive property,  $720X \sim \text{Po}(2160)$

Since  $\lambda = 2160$  is large, we can use a normal distribution to approximate the Poisson distribution, where  $720X \sim \text{N}(2160, 2160)$  approximately.

$$\begin{aligned} & P(720X > 2200) \\ &= P(720X > 2200.5) \text{ (by continuity correction)} \\ &= 0.19176 \\ &\approx 0.192 \end{aligned}$$

(iv) Let  $Y$  be the number of busy working days out of 6 working days.

$$Y \sim \text{B}(6, 0.192)$$

$$\begin{aligned} P(Y = 2) &= 0.23538 \\ &\approx 0.235 \end{aligned}$$

(v) Let  $W$  be the number of busy working days out of 30 working days.

$$np = 30 \times 0.19176 \approx 5.75 > 5$$

$$n(1-p) = 30 \times (1 - 0.19176) \approx 24.2 > 5$$

Since  $np > 5$  and  $n(1-p) > 5$ , we use a normal distribution to approximate the binomial distribution.

$$np(1-p) = 30 \times 0.19176 \times (1 - 0.19176) = 4.65$$

$$W \sim \text{N}(5.75, 4.65)$$

$$\begin{aligned} & P(W < 10) \\ &= P(W < 9.5) \text{ (by continuity correction)} \\ &= 0.95888 \\ &\approx 0.959 \end{aligned}$$