MATHEMATICS

8864/01

11 NOVEMBER 2010

1

$$4x^2 - 2kx + 9 = 0$$

For two real distinct roots to exist,

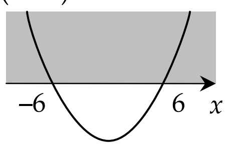
$$b^2 - 4ac > 0$$

$$(-2k)^2 - 4(4)(9) > 0$$

$$4k^2 - 144 > 0$$

$$4(k^2 - 36) > 0$$

$$(k+6)(k-6) > 0$$



$\therefore k < -6$ or $k > 6$

The set of values of k for which the equation $4x^2 - 2kx + 9 = 0$ has two real distinct roots is $\{k \in \mathbb{R} : k < -6 \text{ or } k > 6\}$.

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2 **(i)**

$$\int e^{1-2x}\,dx = \frac{e^{1-2x}}{-2} + c$$

$$= -\frac{1}{2}e^{1-2x} + c$$

(ii)

$$\int \frac{2}{(x+1)^3}\,dx = \frac{2}{-2(x+1)^2} + c$$

$$= -\frac{1}{(x+1)^2} + c$$

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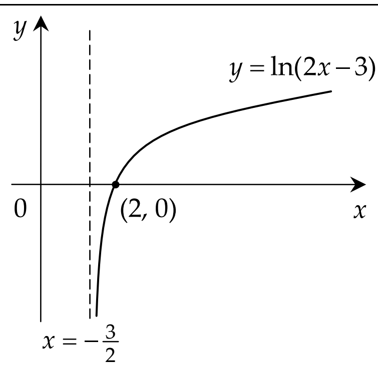
3 **(i)**



(ii)

$$\frac{dy}{dx} = \frac{d}{dx}\ln(2x-3)$$

$$= \frac{1}{2x-3} \times 2$$

$$= \frac{2}{2x-3}$$

(iii) When $x = 3$,

$$y = \ln(2x-3) = \ln(2(3)-3) = \ln 3$$

$$\frac{dy}{dx} = \frac{2}{2x-3} = \frac{2}{2(3)-3} = \frac{2}{3}$$

Gradient of normal $= -\frac{3}{2}$

Equation of normal to the curve is

$$y - y_1 = m(x - x_1)$$

$$y - \ln 3 = -\frac{3}{2}(x-3)$$

$$\frac{3}{2}x + y = \frac{9}{2} + \ln 3$$

$$3x + 2y = 9 + 2\ln 3$$

where $a = 3$ and $b = 2$.

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4 **(i)** From the diagram, $AE = EB$, $AB = CD$ and $AD = BC$.

Given $AE = \frac{5}{8}AB$ and $AB = 2x$ m,

$$AE + EB + BC + CD + DA = 6$$

$$\tfrac{5}{8}AB + \tfrac{5}{8}AB + AD + AB + AD = 6$$

$$\tfrac{9}{4}AB + 2AD = 6$$

$$2AD = 6 - \tfrac{9}{4}AB$$

$$AD = \frac{6 - \frac{9}{4}AB}{2}$$

$$= \frac{6 - \frac{9}{4}(2x)}{2}$$

$$= 3 - \tfrac{9}{4}x \text{ m}$$

(ii)

Area of $ABCD = (2x)(3 - \tfrac{9}{4}x)$

$$= 6x - \tfrac{9}{2}x^2$$

$$AE = \tfrac{5}{8}AB = \tfrac{5}{8}(2x) = \tfrac{5}{4}x$$

Height of $ABE = \sqrt{(AE)^2 - (\tfrac{1}{2}AB)^2}$

$$= \sqrt{\tfrac{25}{16}x^2 - x^2}$$

$$= \sqrt{\tfrac{9}{16}x^2}$$

$$= \tfrac{3}{4}x$$

Area of $ABE = \tfrac{1}{2}(2x)(\tfrac{3}{4}x)$

$$= \tfrac{3}{4}x^2$$

Area of window, $A = 6x - \tfrac{9}{2}x^2 + \tfrac{3}{4}x^2$

$$= 6x - \tfrac{15}{4}x^2 \text{ m}^2$$

(iii) To find the maximum area,

$$\frac{dA}{dx} = 6 - \tfrac{15}{2}x = 0$$

$$\tfrac{15}{2}x = 6 \Rightarrow x = \tfrac{4}{5}$$

$$A = 6(\tfrac{4}{5}) - \tfrac{15}{4}(\tfrac{4}{5})^2 = \tfrac{12}{5} \text{ m}^2$$

To confirm that this value gives us the maximum area, $\frac{d^2A}{dx^2} = -\frac{15}{2} < 0$.

The maximum value of the area is $\frac{12}{5}$ m^2.

5 **(i)** $y = 6 - 4x^3 - 3x^4$

$\frac{dy}{dx} = -4x^2(3) - 3x^3(4)$

$= -12x^2 - 12x^3$

To find the stationary points,

$\frac{dy}{dx} = -12x^2 - 12x^3 = 0$

$-12x^2(1+x) = 0$

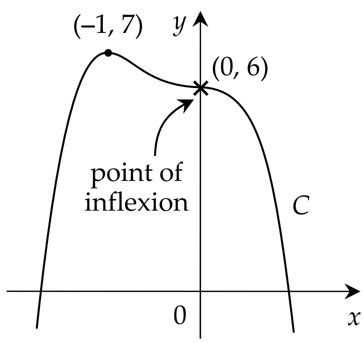
$x = 0$ or $x = -1$

When $x = 0$, $y = 6 - 4(0)^3 - 3(0)^4 = 6$

When $x = -1$, $y = 6 - 4(-1)^3 - 3(-1)^4 = 7$

The coordinates of the stationary points of C are $(0,6)$ and $(-1,7)$.

(ii)



(iii) Using GC, the x-coordinates of the points where C cuts the x-axis are $x = -1.72$ and $x = 0.96$.

(iv) $\int 6 - 4x^3 - 3x^4 \, dx = 6x - \frac{4x^4}{4} - \frac{3x^5}{5} + c$

$= 6x - x^4 - \frac{3x^5}{5} + c$

$\int_{-1}^{\frac{1}{2}} 6 - 4x^3 - 3x^4 \, dx = \left[6x - x^4 - \frac{3x^5}{5} \right]_{-1}^{\frac{1}{2}}$

$= \left[6(\tfrac{1}{2}) - (\tfrac{1}{2})^4 - \frac{3(\frac{1}{2})^5}{5} \right] - \left[6(-1) - (-1)^4 - \frac{3(-1)^5}{5} \right]$

$= \frac{467}{160} - \left(-\frac{32}{5} \right)$

$= \frac{1491}{160}$

6 **(i)** P(both A and B occur)

$= P(A \cap B)$

$= P(A \mid B) \times P(B)$

$= 0.2 \times 0.3$

$= 0.06$

(ii) P(at least one of A and B occurs)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.3 - 0.06$

$= 0.84$

(iii) P(exactly one of A and B occurs)

$= P(A \cup B) - P(A \cap B)$

$= 0.84 - 0.06$

$= 0.78$

7 **(i)** P(fails at both attempts)

$= (1 - 0.9)(1 - 0.7)$

$= 0.03$

(ii) P(passes at second attempt | passes exam)

$= \frac{\text{P(passes at second attempt} \cap \text{passes exam)}}{\text{P(passes exam)}}$

$= \frac{(1-0.7)(0.9)}{0.7 + (1-0.7)(0.9)}$

$= \frac{27}{97}$

(iii) Required probability

$= 0.7 \times 0.7 \times (1-0.7)(0.9) \times \binom{3}{1}$

$= 0.3969$

8 **(i)** Divide the student population into mutually exclusive subgroups, i.e. Year One, Year Two and Year Three students.

Population size $= 1400 + 900 + 700$

$= 3000$

Within each subgroup, we conduct random sampling with a sample size relative to the stratum size as follows:

Number of Year One students to be selected

$= \frac{1400}{3000} \times 60 = 28$

Number of Year Two students to be selected

$= \frac{900}{3000} \times 60 = 18$

Number of Year Three students to be selected

$= \frac{700}{3000} \times 60 = 14$

(ii) Stratified sampling allows us to analyse the amount spent on new clothes by students the three different years separately, which gives a more representative sample.

(iii) $\bar{x} = \frac{\Sigma x}{n} = \frac{10\,450}{50} = 209$

$s^2 = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n} \right)$

$= \frac{1}{50-1}\left(22\,35\,000 - \frac{10\,450^2}{50} \right) = 1039\frac{39}{49}$

The unbiased estimates of the population mean and variance are 209 and $1039\frac{39}{49}$ respectively.

(iv) 1. Since the sample size of 50 is sufficiently large, assume that the amount of money spent by a student follows a normal distribution, based on the Central Limit Theorem.

2. Assume that the unbiased estimate of the variance is sufficiently close to its actual value.

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9 **(i)** Let X be the number of sunflower seeds out of 8 that will germinate, i.e. $X \sim \mathrm{B}(8, 0.7)$.

$P(X = 6) = 0.29648$

≈ 0.296

(ii) $P(X \geq 6) = 1 - \mathrm{P}(X < 6)$

$= 1 - \mathrm{P}(X \leq 5)$

$= 1 - 0.44823$

≈ 0.552

Let Y be the number of sunflower seeds out of 60 that will germinate, i.e. $X \sim \mathrm{B}(60, 0.7)$.

$np = 60 \times 0.7 = 42 > 5$

$n(1-p) = 60 \times (1 - 0.7) = 18 > 5$

Since $np > 5$ and $n(1-p) > 5$, we use a normal distribution to approximate the binomial distribution.

$np(1-p) = 60 \times 0.7 \times (1 - 0.7) = 12.6$

$Y \sim \mathrm{N}(42, 12.6)$

$\mathrm{P}(Y < 40)$

$= \mathrm{P}(Y < 39.5)$ (by continuity correction)

$= 0.24062$

≈ 0.241

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10 Let X be the mass of a component in grams.

$\mathrm{H}_0 : \mu = 15$ vs $\mathrm{H}_1 : \mu \neq 15$

Perform a 2-tail test at the 5% significance level.

Under H_0, $\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$, where

$\mu_0 = 15$, $\bar{x} = 15.25$, $\sigma = 1.2$ and $n = 80$.

Using a t-test, p-value = 0.0624.

Since the p-value = 0.0624 > 0.05, we do not reject H_0 and conclude that there is insufficient evidence at the 5% significance level that the mass of a component has changed.

$\mathrm{H}_0 : \mu = 15$ vs $\mathrm{H}_1 : \mu < 15$

Under H_0, $\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$, where

$\mu_0 = 15$, $\sigma = 1.2$ and $n = 80$.

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 15}{1.2/\sqrt{80}} < -1.6449$$

$$\bar{x} < \frac{-1.6449(1.2)}{\sqrt{80}} + 15$$

$$\bar{x} < 14.779$$

$$\bar{x} < 14.8$$

Since the mean mass is always positive, the set of values required is $\{\bar{x} \in \mathbb{R}^+ : \bar{x} < 14.8\}$.

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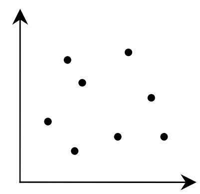
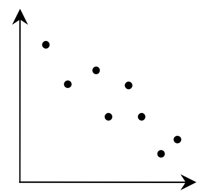
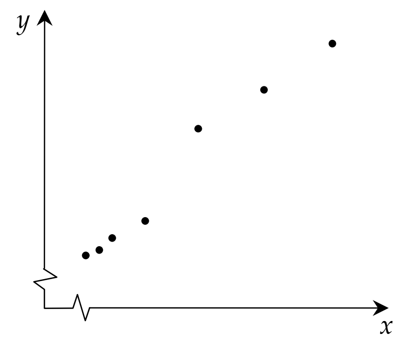
11 **(a)** **(i)**



(ii)



(b) **(i)**



(ii) Product moment correlation coefficient $= 0.987$

(iii) The equation of the regression line of y on x is $y = 0.1069x + 0.5615$

(iv) $y = 0.1069(40) + 0.5615$

$= 4.8375$

An estimate of the monthly earnings of a 40-year-old worker is $4840.

Since $x = 40$ is within the given interval, interpolation gives us a generally reliable result, where we estimate y from the regression line of y on x when x is within the given interval.

(v) The value of m does not change, while the value of c changes to $(c + N)$.

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12 **(i)** Let U be the mass of an unwrapped sweet, i.e. $U \sim \mathrm{N}(40, 3^2)$.

$P(U < 36) = 0.091211$

≈ 0.0912

(ii) Let W be the mass of a wrapped sweet.

$\mathrm{E}(W) = 40 + 4 = 44$

$\mathrm{Var}(W) = 3^2 + 0.5^2 = 9.25$

The mean and variance of the mass of an individual wrapped sweet is 44 and 9.25 respectively.

$\Rightarrow W \sim N(44, 9.25)$

$P(42 < W < 46) = 0.48920$

≈ 0.489

(iii) Let C be the mass of a cardboard tube, where $C \sim N(50, 5^2)$.

$$C + W_1 + W_2 + \ldots + W_{12} \sim N\begin{pmatrix} 50 + 12(44), \\ 5^2 + 12(9.25) \end{pmatrix}$$

$$C + W_1 + W_2 + \ldots + W_{12} \sim N(578, 136)$$

$P(C + W_1 + W_2 + \ldots + W_{12} > 600)$

$= 0.029615$

≈ 0.0296

(iv) Let R be the mass of a tube of sweets produced by a rival company, where $R \sim N(\mu, \sigma^2)$.

$P(R < 450) = 0.05$

$P\left(Z < \dfrac{450 - \mu}{\sigma}\right) = 0.05$

$P(R > 550) = 0.08$

$P\left(Z > \dfrac{550 - \mu}{\sigma}\right) = 0.08$

$P\left(Z < \dfrac{550 - \mu}{\sigma}\right) = 0.92$

$\dfrac{450 - \mu}{\sigma} = -1.6449$

$-1.6449\sigma + \mu = 450$

$\dfrac{550 - \mu}{\sigma} = 1.4051$

$1.4051\sigma + \mu = 550$

$1.4051\sigma - (-1.6449)\sigma = 550 - 450$

$3.0499\sigma = 100$

$\sigma = 32.788$

$\sigma^2 = 1075.0$

≈ 1080

$\mu = 450 + 1.6449(32.8) \approx 504$

The mean and variance of the distribution is 504 and 1080 respectively.