ADDITIONAL MATHEMATICS

4038/02

29 OCTOBER 2010

1

$$3\cot^2\theta + 10\operatorname{cosec}\theta = 5$$

$$3(\operatorname{cosec}^2\theta - 1) + 10\operatorname{cosec}\theta - 5 = 0$$

$$3\operatorname{cosec}^2\theta - 3 + 10\operatorname{cosec}\theta - 5 = 0$$

$$\frac{3}{\sin^2\theta} - 8 + \frac{10}{\sin\theta} = 0$$

$$3 - 8\sin^2\theta + 10\sin\theta = 0$$

$$8\sin^2\theta - 10\sin\theta - 3 = 0$$

$$(2\sin\theta - 3)(4\sin\theta + 1) = 0$$

$2\sin\theta = 3$ OR $4\sin\theta = -1$

$\sin\theta = \frac{3}{2}$ (rej.) $\quad \sin\theta = -\frac{1}{4}$

Basic angle $\theta = \sin^{-1}\frac{1}{4} = 14.478°$

$\theta = 180° + 14.478°$ OR $\theta = 360° - 14.478°$

$\approx 194.5° \quad \approx 345.5°$

2 **(i)** Considering similar triangles PQR and PWV,

$$\frac{PQ}{QR} = \frac{PW}{WV}$$

$$\frac{8}{12} = \frac{8-y}{x}$$

$$\frac{8x}{12} = 8 - y$$

$$y = 8 - \frac{2x}{3}$$

(ii) $A = xy$

$$= x\left(8 - \frac{2x}{3}\right)$$

$$= 8x - \frac{2x^2}{3}$$

(iii) $\frac{\mathrm{d}A}{\mathrm{d}x} = 8 - \frac{4x}{3} = 0$

$$8 = \frac{4x}{3}$$

$$(8)(3) = 4x$$

$$x = \frac{24}{4} = 6$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -\frac{4}{3} < 0$$

When $x = 6$, $A = 8(6) - \frac{2(6)^2}{3} = 24$.

Since $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} < 0$, the maximum value of A is 24.

3

$32^x \times 2^y = 1$

$2^{5x} \times 2^y = 2^0$

$2^{5x+y} = 2^0$

$5x + y = 0$

$y = -5x$

$3^{x-12} \div 27^y = 81^{\frac{1}{x}}$

$3^{x-12} \div 3^{3y} = 3^{\frac{4}{x}}$

$3^{x-12-3y} = 3^{\frac{4}{x}}$

$x - 12 - 3y = \frac{4}{x}$

$x^2 - 12x - 3xy = 4$

$x^2 - 12x - 3x(-5x) = 4$

$x^2 - 12x + 15x^2 - 4 = 0$

$16x^2 - 12x - 4 = 0$

$4x^2 - 3x - 1 = 0$

$(4x+1)(x-1) = 0$

$x = -\frac{1}{4}$ OR $x = 1$

$y = -5\left(-\frac{1}{4}\right) \quad y = -5(1)$

$= \frac{5}{4} \quad = -5$

The solutions are $x = -\frac{1}{4}$, $y = \frac{5}{4}$ and $x = 1$, $y = -5$.

4 **(i)** Constant term in $\left(x - \frac{k}{x^3}\right)^8 = \binom{8}{2}x^6\left(-\frac{k}{x^3}\right)^2$

$$= 28x^6\left(\frac{k^2}{x^6}\right)$$

$$= 28k^2 = 7$$

$$k^2 = \frac{1}{4} \Rightarrow k = \pm\frac{1}{2}$$

Since k is positive, $k = \frac{1}{2}$.

(ii) x^{-4} term in $\left(x - \frac{k}{x^3}\right)^8 = \binom{8}{3}x^5\left(-\frac{1}{2x^3}\right)^3$

$$= 56x^5\left(-\frac{1}{8x^9}\right) = -\frac{7}{x^4}$$

Constant term in $(1+x^4)\left(x - \frac{1}{2x^3}\right)^8$

$= \text{Constant in } (1+x^4) \times \text{Constant in } \left(x - \frac{1}{2x^3}\right)^8$

$+ x^4 \text{ term in } (1+x^4) \times x^{-4} \text{ term in } \left(x - \frac{1}{2x^3}\right)$

$= (1)(7) + (x^4)\left(-\frac{7}{x^4}\right)$

$= 7 - 7$

$= 0$

Since the constant term is zero, there is no constant term.

5 **(i)** $y = 5 - e^{2x}$

When $x = 0$, $y = 5 - e^{2(0)} = 5 - 1 = 4$

When $y = 0$, $0 = 5 - e^{2x}$

$$5 = e^{2x}$$
$$2x = \ln 5$$
$$x = \frac{\ln 5}{2}$$

$\therefore A(0,4)$ and $B\left(\frac{\ln 5}{2}, 0\right)$

Let P and Q denote the points $(0,k)$ and $(\ln 5, k)$ respectively.

Consider the similar triangles AOB and APQ.

$$\frac{4-0}{\frac{\ln 5}{2}} = \frac{4-k}{\ln 5}$$
$$4\ln 5 = (4-k)\frac{\ln 5}{2}$$
$$8 = 4 - k$$
$$\therefore k = -4$$

(ii)
$$x = \ln\sqrt{9-x}$$
$$e^x = \sqrt{9-x}$$
$$e^{2x} = 9 - x$$
$$5 - e^{2x} = 5 - (9 - x)$$
$$= x - 4$$

The equation of the straight line is $y = x - 4$.

6 **(i)** Let k denote $B\hat{X}A$.

$Y\hat{X}A = B\hat{X}A = k$ (given XA bisects $Y\hat{X}B$)

$A\hat{C}X = Y\hat{X}A = k$ (tangent chord theorem)

$A\hat{B}X = A\hat{C}X = k$ (angle in the same segment)

Since $B\hat{X}A = A\hat{B}X = k$, ABX is an isosceles triangle. $\therefore AX = AB$

(ii) $A\hat{C}B = A\hat{X}B = k$ (angle in the same segment)

Since $A\hat{C}X = A\hat{C}B = k$, and D lies on AC, CD bisects $X\hat{C}B$.

(iii) In triangles CDX and CBA,

$X\hat{C}D = A\hat{C}B$ (since CD bisects $X\hat{C}B$)

$C\hat{X}D = C\hat{A}B$ (angle in the same segment)

$C\hat{D}X = C\hat{B}A$ (angle sum of a triangle)

By the AAA property, triangles CDX and CBA are similar.

7 **(i)**
$$y = \sqrt{2x+5}$$
$$\frac{dy}{dx} = \frac{2}{2\sqrt{2x+5}}$$
$$= \frac{1}{\sqrt{2x+5}}$$

When $x = 2$, $\frac{dy}{dx} = \frac{1}{\sqrt{2(2)+5}} = \frac{1}{3}$

x-coordinate of $Q = -\frac{5}{2} \Rightarrow Q\left(-\frac{5}{2}, 0\right)$

$$y - 0 = \frac{1}{3}\left(x - \left(-\frac{5}{2}\right)\right) = \frac{1}{3}x + \frac{5}{6}$$

The equation of QR is $y = \frac{1}{3}x + \frac{5}{6}$.

(ii) Area of QPS

$$= \int_{-\frac{5}{2}}^{2} \sqrt{2x+5}\ dx$$
$$= \left[\frac{\sqrt{(2x+5)^3}}{2 \times \frac{3}{2}}\right]_{-\frac{5}{2}}^{2}$$
$$= \frac{\sqrt{(2(2)+5)^3}}{2\times\frac{3}{2}} - \frac{\sqrt{(2(-\frac{5}{2})+5)^3}}{2\times\frac{3}{2}}$$

$= 9$ square units

Considering QR, when $x = 2$,

$$y = \frac{1}{3}(2) + \frac{5}{6} = \frac{3}{2}$$

Area of QST = Area of QSR

$$= \frac{1}{2}\left(2 - \left(-\frac{5}{2}\right)\right)\left(\frac{3}{2}\right)$$
$$= \frac{27}{8} \text{ square units}$$

Total area $= 9 + \frac{27}{8} = \frac{99}{8}$ square units

8 **(i)**

$$v_P = \int a_P\ dt$$
$$= \int 1.5\ dt$$
$$= 1.5t + c$$

OR

$$v_Q = \int a_Q\ dt$$
$$= \int 1 + \frac{t}{2}\ dt$$
$$= t + \frac{t^2}{2(2)} + c$$
$$= t + \frac{t^2}{4} + c$$

When $t = 0$, $c = 9$ | When $t = 0$, $c = 0$

$v_P = 1.5t + 9$ | $v_Q = t + \frac{t^2}{4}$

(ii)

$$s_P = \int v_P\ dt$$
$$= \int 9 + 1.5t\ dt$$
$$= 9t + \frac{1.5t^2}{2} + c$$
$$= 9t + \frac{3t^2}{4} + c$$

$$s_Q = \int v_Q\ dt$$
$$= \int t + \frac{t^2}{4}\ dt$$
$$= \frac{t^2}{2} + \frac{t^3}{12} + d$$

Since both particles started at the origin, when $t = 0$, $s_P = c = 0$ and $s_Q = d = 0$.

The distance travelled by P is $s_P = 9t + \frac{3t^2}{4}$.

The distance travelled by Q is $s_Q = \frac{t^2}{2} + \frac{t^3}{12}$.

(iii) $s_P = s_Q$

$$9t + \frac{3t^2}{4} = \frac{t^2}{2} + \frac{t^3}{12}$$
$$108t + 9t^2 = 6t^2 + t^3$$
$$t^3 - 3t^2 - 108t = 0$$
$$t(t^2 - 3t - 108) = 0$$
$$t(t+9)(t-12) = 0$$
$$t = 0,\ t = -9 \text{ OR } t = 12$$
$$s = 9t + \frac{3t^2}{4}$$
$$= 216 \text{ m}$$

(iv) $v_P = 9 + 1.5t = 27$ m/s

$$v_Q = t + \frac{t^2}{4} = 48 \text{ m/s}$$

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9 **(i)** $AB = BC$

$$\sqrt{(8-0)^2 + (14-5)^2} = \sqrt{(k-8)^2 + (15-14)^2}$$
$$145 = k^2 - 16k + 64 + 1$$
$$k^2 - 16k - 80 = 0$$
$$(k-20)(k+4) = 0$$
$$k = 20 \quad \text{OR} \quad k = -4$$

Since k is positive, we have $k = 20$.

(ii) Since $AD = CD$, D lies on the perpendicular bisector of AC.

Let M denote the midpoint of AC.

$$M\left(\frac{0+20}{2}, \frac{5+15}{2}\right) \Rightarrow M(10,10)$$

Gradient of $BM = \frac{14-10}{8-10} = \frac{4}{-2} = -2$

$$y - 10 = -2(x-10)$$
$$y = -2x + 20 + 10$$
$$= -2x + 30$$

The equation of the line BD is $y = -2x + 10$.

To find the coordinates of D, let $y = 0$.

$$0 = -2x + 30$$
$$2x = 30 \Rightarrow x = 15$$
$$\therefore D(15,0)$$

(iii) $BM = \sqrt{(8-10)^2 + (14-10)^2} = \sqrt{20} = 2\sqrt{5}$

$MD = \sqrt{(10-15)^2 + (10-0)^2} = \sqrt{125} = 5\sqrt{5}$

$$\frac{\text{Area of } \Delta ABC}{\text{Area of quadrilateral } ABCD}$$
$$= \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ABC + \text{Area of } \Delta ADC}$$
$$= \frac{\frac{1}{2}(AC)(BM)}{\frac{1}{2}(AC)(BM) + \frac{1}{2}(AC)(MD)}$$
$$= \frac{\frac{1}{2}(AC)(2\sqrt{5})}{\frac{1}{2}(AC)(2\sqrt{5}) + \frac{1}{2}(AC)(5\sqrt{5})}$$
$$= \frac{\frac{1}{2}(AC)(2\sqrt{5})}{\frac{1}{2}(AC)(7\sqrt{5})}$$
$$= \frac{2}{7}$$

$\therefore$ Area of ΔABC

$$= \frac{2}{7} \times \text{Area of quadrilateral } ABCD$$

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10 **(i)**

$$\frac{3x^2 + 4x - 20}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx + C}{x^2 + 4}$$
$$3x^2 + 4x - 20 = A(x^2+4) + (Bx + C)(2x+1)$$

Let $x = -0.5$

$$3(-0.5)^2 + 4(-0.5) - 20 = A((-0.5)^2 + 4) + 0$$
$$-\frac{85}{4} = \frac{17}{4}A$$
$$A = -\frac{85}{4} \div \frac{17}{4} = -5$$
$$\Rightarrow 3x^2 + 4x - 20 = -5(x^2+4) + (Bx + C)(2x+1)$$

Let $x = 0$

$$-20 = -5(4) + 3C$$
$$3C = -5(4) + 20 = 0$$
$$C = 0$$
$$\Rightarrow 3x^2 + 4x - 20 = -5(x^2+4) + (Bx)(2x+1)$$

Let $x = 1$

$$3(1)^2 + 4(1) - 20 = -5((1)^2 + 4) + (B(1))(2(1)+1)$$
$$-13 = -25 + 3B$$
$$3B = -13 + 25 = 12$$
$$B = 4$$

$\therefore \frac{3x^2 + 4x - 20}{(2x+1)(x^2+4)} = \frac{-5}{2x+1} + \frac{4x}{x^2+4}$, where $A = -5,\ B = 4,\ C = 0$.

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x^2+4)$$
$$= \frac{1}{x^2+4} \times 2x$$
$$= \frac{2x}{x^2+4}$$

(iii)

$$\int \frac{3x^2+4x-20}{(2x+1)(x^2+4)}\,dx$$

$$= \int \frac{-5}{2x+1} + \frac{4x}{x^2+4}\,dx$$

$$= \int \frac{-5}{2x+1}\,dx + \int \frac{4x}{x^2+4}\,dx$$

$$= -5\int \frac{1}{2x+1}\,dx + 2\int \frac{2x}{x^2+4}\,dx$$

$$= -5\frac{\ln(2x+1)}{2} + 2\ln(x^2+4) + c$$

$$= -\frac{5}{2}\ln(2x+1) + 2\ln(x^2+4) + c,\ x > -\frac{1}{2}$$

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11 **(i)** $A\left(\frac{\pi}{2}, 5\right)$ $B\left(\frac{3\pi}{2}, -1\right)$ $C(\pi, -4)$

(ii) $4\cos x = 2 + 3\sin x$

$4\cos x - 3\sin x = 2$

$R\cos(x+\alpha) = 2$

$R = \sqrt{4^2+3^2} = 5$

$\alpha = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{3}{4} = 0.64350 \approx 0.644$

Now, $5\cos(x+0.644) = 2$

$\cos(x+0.644) = 0.4$

where $\alpha = 0.644,\ k = 0.4$.

(iii) $\cos(x+0.644) = 0.4$

Basic angle $(x+0.644) = \cos^{-1}0.4$

$= 1.1593$

$x+0.644 = 1.1593$ OR $x+0.644 = 2\pi - 1.1593$

$x \approx 0.516$ $\qquad$ $x \approx 4.48$

x-coordinate of $D = 0.516$

x-coordinate of $E = 4.48$
